(1)

Workout Exam Num. Math 1, June 13, 2016

1 a Polynomial Interpolation of a set of data points (x_k, y_k), $k=0,\dots,n$ can be written in terms of Lagrange characteristic polynomials as

$$\Pi_n(x) = \sum_{k=0}^{n} y_k \varphi_k(x)$$

Give the general expression for $\varphi_k(x)$ and show that $\Pi_n(x_j) = y_j$

$\boxed{2}$

Answer

$\varphi_k(x)$ is a polynomial s.t. $\varphi_k(x_j) = \delta_{kj}$

Def $\hat{\varphi}_k(x) = \prod_{\substack{l=0 \\ l\neq k}}^{n} (x - x_l)$

Then $\varphi_k(x) = \dfrac{\hat{\varphi}_k(x)}{\hat{\varphi}_k(x_k)} = \prod_{\substack{l=0 \\ l\neq k}}^{n} \dfrac{(x-x_l)}{(x_k - x_l)}$

$$\Pi_n(x_j) = \sum_{k=0}^{n} y_k \varphi_k(x_j) = \sum_{k=0}^{n} y_k \delta_{kj} = y_j$$

1 b For the numerical integration of $\int_{-1}^{1} f(x)\,dx$, Simpsons rule is defined as

$$\frac{1}{3} f(-1) + \frac{4}{3} f(0) + \frac{1}{3} f(1) \quad *$$

i Why is the sum of the coefficients 2 ?

$\boxed{2}$

Ans: because it should at least be exact for a constant function $f(x) \equiv c$

Integr : $\int_{-1}^{1} c\,dx = 2c$

Simpson : $\frac{1}{3}c + \frac{4}{3}c + \frac{1}{3}c = 2c$

ii Show that the interpolation polynomial used to derive this method is:

$$\Pi_2(x) = \frac{1}{2} f(-1)\, x(x-1) + f(0)(1-x^2)$$
$$+ \frac{1}{2} f(1)\, x(x+1)$$

3

Answer: the polynomial associated to $f(-1)$ should be zero at $x=0$ and $x=1$
hence a factor times $x(x-1)$
the factor should be such that this polyn becomes 1 for $x=-1$ → hence $\frac{1}{2}$

Likewise for the other coefficients

iii Derive Simpson's rule from this interpolation polynomial.

Answer: Integrate $\Pi_2(x)$ over $[-1, 1]$
In fact we have to integrate

3

$$\frac{1}{2}\int_{-1}^{1} x(x-1)\,dx = \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_{-1}^{1} = \frac{1}{3}$$

$$\int_{-1}^{1} 1-x^2\,dx = x - \frac{1}{3}x^3\Big|_{-1}^{1} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\frac{1}{2}\int x(x+1) = \frac{1}{2}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_{-1}^{1} = \frac{1}{3}$$

Hence the rule is $\frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)$

iv: Show that the error is

Niet optentemen

Answer: The error in the integration rule is the integral of the interpolation error.

Here interpolation error:

$$\frac{1}{6} x(x^2-1) f'''(\xi(x))$$

Integration: $\frac{1}{6}\int_{-1}^{1} \underbrace{x(x^2-1)}\, f'''(\xi(x))\, dx =$

problematisch $\Rightarrow$

$\left\{ \begin{array}{l} \text{observe odd function} \\ \text{so if } f''' \text{ would be constant then error} \\ \text{zero} \longrightarrow \text{replace } f'''(\xi(x)) \text{ by} \\ \underbrace{f'''(\xi(0))}_{\text{constant}} + x\, f^{IV}(\underbrace{\eta(\xi(x))}_{\pi(x)}) \end{array} \right\}$

$$= \frac{1}{6}\int_{-1}^{1} \underbrace{x^2(x^2-1)}_{\text{definite}} f^{IV}(\pi(x))\, dx =$$

$$= \frac{1}{6} f^{IV}(\mu(x)) \int_{-1}^{1} x^2(x^2-1) = \frac{1}{6} f^{IV}(\mu(x)) \underbrace{\left(\frac{1}{5}x^5 - \frac{1}{3}x^3\right)\Big|_{-1}^{1}}_{\frac{2}{5}-\frac{2}{3}=-\frac{4}{15}}$$

$$= -\frac{2}{45} f^{IV}(\mu(x))$$

iv The interval $[a,b]$ can be transformed to $[-1,1]$ by the linear function $x(s)$ s.t. $x(-1)=a$ and $x(1)=b$

$\boxed{3}$ Give $x(s)$ and derive from that Simpsons rule on the interval $[a,b]$

Answer

$$x(s) = \alpha s + \beta$$

$$(-1) = -\alpha + \beta = a$$

$$(1) = \alpha + \beta = b$$

$$\Rightarrow \quad \beta = \frac{a+b}{2}$$

$$\alpha = \frac{b-a}{2}$$

$$x(s) = \frac{b-a}{2} s + \frac{a+b}{2}$$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2} s + \frac{a+b}{2}\right) \frac{b-a}{2}\,ds$$

$$= \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2} s + \frac{a+b}{2}\right) ds =$$

Simpson rule

$$= \frac{b-a}{2}\left(\frac{1}{3} f(a) + \frac{4}{3} f\left(\frac{a+b}{2}\right) + \frac{1}{3} f(b)\right) =$$

$$= (b-a)\left(\frac{1}{6} f(a) + \frac{2}{3} f\left(\frac{a+b}{2}\right) + \frac{1}{6} f(b)\right)$$

2 Consider the fixed point iteration $x^{(n+1)} = \phi(x^{(n)})$ where

$$\phi_1(x_1, x_2) = x_1 + \frac{1}{2} x_2$$
$$\phi_2(x_1, x_2) = \frac{1}{2} x_2 + \frac{1}{2} x_1 - \frac{1}{2} x_1^3$$

a Show that the origin is a fixed point of this iteration

$\boxed{1}$ Answer indeed $\phi(0) = 0$

b Compute the Jacobian matrix of ϕ

Answer

$\boxed{3}$ $$J_\phi(x) = \begin{bmatrix} \phi_{1x_1} & \phi_{1x_2} \\ \phi_{2x_1} & \phi_{2x_2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} - \frac{3}{2} x_1^2 & \frac{1}{2} \end{bmatrix}$$

c Show that $e^{(n+1)} \approx J_\phi(0) e^{(n)}$

$\boxed{4}$ Answer:
$$\left.\begin{matrix} x^{(n+1)} = \phi(x^{(n)}) \\ 0 = \phi(0) \end{matrix}\right\} \underbrace{(x^{(n+1)} - 0)}_{e^{(n+1)}} = \phi(x^{(n)}) - \phi(0)$$
$$e^{(n+1)} \approx J_\phi(0)(x^{(n)} - 0) = J_\phi(0) e^{(n)}$$

d Will the fixed point method converge to 0 when starting close to it?

Answer:
Compute the eigenvalues of $J_\phi(0)$
They should all be <1 in abs. value.

$\boxed{3}$ $$J_\phi(0) \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \det \begin{bmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{bmatrix} = 0$$

$$(1-\lambda)\left(\frac{1}{2} - \lambda\right) - \frac{1}{4} = 0$$
$$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{4} = 0$$
$$\frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - 1}}{2} = \frac{3}{4} \pm \frac{1}{4}\sqrt{5} = \frac{1}{4}(3 \pm \sqrt{5}) = \begin{cases} \frac{1}{4}(3+\sqrt{5}) > \frac{5}{4} > 1 \\ \frac{1}{4}(3-\sqrt{5}) < 1 \end{cases}$$

diverge

so the method will <u>not</u> converge to 0 d Show that also $(1,0)$ is a fixed point

Answer.

$$\phi(1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

e Is the fixed point method converging in the neighbourhood of this fixed point?

$$J\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & \frac{1}{2} \\ -1 & \left(\frac{1}{2}-\lambda\right) \end{bmatrix} = 0$$

$$(1-\lambda)\left(\tfrac{1}{2}-\lambda\right) + \frac{1}{2} = 0$$

$$\lambda^2 - \frac{3}{2}\lambda + 1 = 0$$

$$\lambda_\pm = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - 4}}{2} = \frac{3}{4} \pm \frac{1}{4}\sqrt{-7} = \frac{3}{4} \pm \frac{\sqrt{7}}{4} i$$

$$|\lambda_\pm| = \frac{1}{4}\sqrt{9+7} = 1$$

hence it is not converging.

3 Consider the problem $Ax = b$

a) assume that b is perturbed to $b + \Delta b$ resulting in a perturbed solution $x + \Delta x$

Show that

$$\frac{\|\Delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \, \frac{\|\Delta b\|}{\|b\|}$$

$\boxed{4}$

where the matrix and vector norm are related by

$$\|A\| = \max_x \frac{\|Ax\|}{\|x\|}$$

Answer
From the matrix norm def it follows that for general x

$$\frac{\|Ax\|}{\|x\|} \leq \|A\|$$

$$\rightarrow \|Ax\| \leq \|A\| \, \|x\|$$

Hence we have $\|b\| = \|Ax\| \leq \|A\| \, \|x\|$ $\quad *$

Similarly $\|x\| = \|A^{-1} b\| \leq \|A^{-1}\| \, \|b\|$

Moreover we have

$$\begin{aligned} A(x + \Delta x) &= b + \Delta b \\ A x &= b \\ \hline A \Delta x &= \Delta b \end{aligned}$$

$$\rightarrow \quad \Delta x = A^{-1} \Delta b \Rightarrow \|\Delta x\| \leq \|A^{-1}\| \, \|\Delta b\|$$

From $*$ we have

$$\frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$$

So $\|\Delta x\| \dfrac{1}{\|x\|} \leq \|A^{-1}\| \, \|\Delta b\| \, \dfrac{\|A\|}{\|b\|}$

from which the result follows.

b Let $A = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Consider ~~following~~ iteration methods for ~~this problem~~

5 Let P be the lower triangular part of A.
and define an iteration by

$$P x^{(n+1)} = (P - A) x^{(n)} + b$$

Define $P = \begin{bmatrix} 2 & & \\ -1 & 2 & \\ & -1 & 1 \end{bmatrix}$, $P-A = U = \begin{bmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ 0 & 0 & 0 \end{bmatrix}$

$$P x^{(n+1)} = +U x^{(n)} + b$$

Show that the error $e^{(n)} = x^{(n)} - x$ satisfies $e^{(n+1)} = B e^{(n)}$ where $B = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

Answer

$$B = P^{-1} U$$

From $Ax = b$ it follows for arbitrary P

$$Px = Px - Ax + b$$
$$= (P - A)x + b$$

$$P x^{(n+1)} = +U x^{(n)} + b$$
$$P x = +U x + b$$
$$P e^{(n+1)} = +U e^{(n)}$$
$$e^{(n+1)} = P^{-1} U e^{(n)}$$

$$P B = +U$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \\ & -1 & 1 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 0 & 0 & \frac{1}{2} & 1 \\ & -1 & 1 & 0 & 0 & 0 \end{array}\right]$$

Rework to identity

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} \end{array}\right]$$

c Show that ~~this~~ Gauss Seidel applied to this problem converges

3 For convergence

$$\rho(B) < 1$$

We know that $\rho(B) \le \|B\|_\infty = \max_{1 \le i \le 3} \sum_{j=1}^{3} |b_{ij}| = \frac{3}{4}$

Alternative: compute the eigenvalues: one eigenv is zero

3c continued

$$\det \begin{bmatrix} (\frac{1}{4}-\lambda) & \frac{1}{2} \\ \frac{1}{4} & (\frac{1}{2}-\lambda) \end{bmatrix} = 0$$

$$(\frac{1}{4}-\lambda)(\frac{1}{2}-\lambda) - \frac{1}{8} = 0$$

$$-\frac{3}{4}\lambda + \lambda^2 = 0$$

$$\lambda(\lambda - \frac{3}{4}) = 0$$

$\rightarrow \rho(B) = \frac{3}{4}$ convergence

3d From the previous we know that $e^{(n)} \to 0$ for $n \to \infty$. It wil however also converge ~~to a certain~~ quickly to a one dimensional subspace. ~~which one~~ Compute the generating vector of this space

Answer

This is a power iteration. The largest eigenvalue is $\frac{3}{4}$

$\boxed{3}$

The associated one dimensional subspace is spanned by the corresponding eigenvector

$$\begin{bmatrix} -\frac{3}{4} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$v_2 = v_3 = 1 \rightarrow v_1 = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3}$$

$$v = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

4 Consider the elliptic problem

$$u - \frac{\partial}{\partial x}\left((1+x)\frac{\partial u}{\partial x}\right) - \frac{\partial^2 u}{\partial y^2} = e^{x+y}$$

on the unit square, where on the boundaries we prescribe ~~the~~

$$u(x,0) = 0$$
$$u(0,y) = 0$$
$$u(x,1) = x$$
$$u(1,y) = y$$

a Show that in general

$$\left[\frac{d}{dx}\left(f(x)\frac{du}{dx}\right)\right]_{x=x_m} = \frac{f(x_m+\frac{1}{2}h)(u(x_{m+1})-u(x_m)) - f(x_m-\frac{1}{2}h)(u(x_m)-u(x_{m-1}))}{h^2} + O(h^2)$$

[4]

in ~~three~~ steps.

(i) First show that
In general

$$\frac{dw}{dx}(x) = \frac{w(x+\frac{1}{2}h) - w(x-\frac{1}{2}h)}{h} - \frac{1}{24}h^2\frac{d^3w}{dx^3}(x) + O(h^3)$$

[2]

(ii) Second, show that (using (i))

$$f(x_m+\tfrac{1}{2}h)\frac{du}{dx}(x_m+\tfrac{1}{2}h) = f(x_m+\tfrac{1}{2}h)\frac{u(x_{m+1})-u(x_m)}{h} - \frac{1}{24}h^2 f(x_m+\tfrac{1}{2}h)\frac{d^3u}{dx^3}(x_m+\tfrac{1}{2}h) + O(h^3)$$

[2]

iii Third, substituting for w in (i) the result of (ii) ~~into~~ to obtain the result.

Answer i

Taylor: $w(x \pm \frac{1}{2}h) = w(x) \pm \frac{1}{2}h\, w'(x) + \frac{1}{8}h^2 w''(x) \pm \frac{1}{48}h^3 w'''(x) + O(h^4)$

[4] $$\frac{w(x+\frac{1}{2}h) - w(x-\frac{1}{2}h)}{h} = w'(x) + \frac{1}{24}h^2 w'''(x) + O(h^3)$$

Answer ii

[2] Set $w(x) = u(x_m + \frac{1}{2}h)$ in the above and multiply by $f(x_m + \frac{1}{2}h)$.

Answer iii

Set $w(x+\frac{1}{2}h) =$ rhs in ii

[3] then we get the desired expression

Check whether they have this!

$$-\frac{1}{24}h^2\left(f(x_m+\tfrac{1}{2}h)\frac{d^3u}{dx^3}(x_m+\tfrac{1}{2}h) - f(x_m-\tfrac{1}{2}h)\frac{d^3u}{dx^3}(x_m-\tfrac{1}{2}h)\right)/h$$
$$+\ O(h^2)$$
$$= -\frac{1}{24}h^2\frac{d}{dx}\left(f(x)\frac{d^3u}{dx^3}\right) + O(h^2)$$

hence the result holds.

b Using the above give the discretization at a general gridpoint (m,n) of the above equation on a grid with mesh size $h = \frac{1}{M+1}$ in both directions.

~~Also~~ (i) at a general gridpoint.

(ii) give also the discretized boundary conditions

Answer i

[4] $$V_{m,n} - \frac{(1+(m+\frac{1}{2})h)(V_{m+1,n} - \cancel{2}V_{m,n}) - (1+(m-\frac{1}{2})h)(V_{m,n} - V_{m-1,n})}{h^2}$$
$$- \frac{V_{m,n+1} - 2V_{m,n} + V_{m,n-1}}{h^2} = e^{(m+n)h} \qquad m = 1,..,M \quad n = 1,\dots,M$$

[2] ii $V_{m,0} = 0$, $V_{m,M+1} = mh$ $\quad m = 1,\dots,M$

$V_{0,n} = 0$, $V_{M+1,n} = nh$ $\quad n = 1,\dots,M$